THE CHALLENGE: TRAVEL-TIME FOR A SINGLE ORIGIN-DESTINATION

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The BRISA problem was to calculate the real time travel information using the Via-Verde (VV) technology (floating data) and automatic data for the area defined by the triangle A5 Lisboa-Cascais and A9 Estádio-Queluz.

To solve the problem we will be able to make use of the large amounts of traffic data that will soon become accessible electronically in real time; indeed there are two different information sources producing different data for the same object and context. There is a tremendous demand for simple information concerning just the travel-time for a single origin-destination and service level for a motorway section. It was suggested that we should use a Hybrid Traffic Flow Modelling to cope with this challenge.

There are some natural scales that bound the range of possible models that can be constructed. Namely

- time-scale: $t_0 \sim O(10^3)$s
- vehicle separation time: $T = k^{-1} \sim O(10)$s,
- drivers reaction time: $\Delta t \sim O(1)$s,
- typical distance between cars: $h \sim O(10)$m,
- typical car velocity: $u \sim O(10)$m/s,
- length scale: $L \sim O(10^3)$m.

Note that for N given cars we have typically $Nh \sim 10^3$ m, but the journey length is $\sim O(10^5)$m. The meaning of these scales will become clear in what follows.

1. INTRODUCTION

Let us briefly outline the content of this report. The basic model approaches for vehicle traffic are described in section 2; the microscopic follow-the-leader models are explain in subsection 2.1 and the macroscopic traffic models in subsection 2.2, see [1] and references therein. In subsection 2.1 we give an explicit solution for the dynamics of the cars using "Pipes rule". Using this rule we also propose a statistical physics model which will allow us to estimate, in a simple manner, the density of the cars and the mean velocity of the car flow. The same rule is used, as suggested by BRISA, for the construction of a hybrid partial differential equation in section 3.

2. BASIC MODELLING APPROACHES

2.1 MICROSCOPIC FOLLOW-THE-LEADER MODELS

Early microscopic traffic models were proposed by Reuschel (1950) and Pipes (1953) [1,2]. Microscopic traffic models assume that the acceleration $dv_i(t)/dt$ of a vehicle i is given by the behaviour of the next vehicle ahead $i+1$. Therefore we can write the following general model of driver behaviour:

$$\frac{dv_i(t)}{dt} = \frac{{v_i}^0 + \xi_i(t) - v_i(t)}{\tau_i} + f_{i,i+1}(t). \tag{1}$$

Here $\xi_i(t)$ corresponds to a fluctuation term and, $f_{i,i+1}(t) \leq 0$ describes the normally repulsive effect of the vehicle $i+1$, which is generally a function of the relative velocity $\Delta v_i(t) = v_i(t) - v_{i+1}(t)$, the velocity $v_i(t)$ of vehicle i due to the velocity-dependent safe distance kept to the vehicle in front, the headway distance $d_i(t) = x_{i+1}(t) - x_i(t)$ or the clearance distance $s_i(t) = d_i(t) - l_{i+1}$, with l_i meaning the length of vehicle. Consequently, for identically behaving

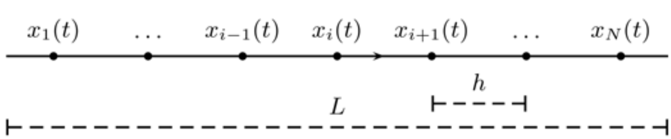

Figure 1: Diagram for the distribution of vehicles.

vehicles with $v_i^0 = v_0$, $\tau_i = \tau$, and $f_{i,i+1}(t) = f$, we would have

$$f_{i,i+1}(t) = f(s_i(t), v_i(t), \Delta v_i(t)). \tag{2}$$

If we neglect fluctuations and introduce the traffic-dependent velocity

$$v(s_i, v_i, \Delta v_i) = v_0 + \tau f(s_i, v_i, \Delta v_i) \tag{3}$$

to which driver tries to adapt, we can considerably simplify the generalized behavioral model to

$$\frac{dv_i(t)}{dt} = \frac{v(s_i, v_i, \Delta v_i) - v_i}{\tau}. \tag{4}$$

Models of the type of equation (4) are called follow-the-leader models. One of the simplest examples results from the assumption that the clearance distance is given by the velocity-dependent safe distance $s^*(v_i) = s' + Tv_i$, where T has the meaning of the (effective) safe time clearance (also called separation time). This implies $s_i(t) = s^*(v_i(t))$ or, after differentiation with respect to time,

$$\frac{dv_i(t)}{dt} = \frac{1}{T}\left(v_{i+1}(t) - v_i(t)\right). \tag{5}$$

A simple Laplace transform calculation gives

$$v_i(t) = \frac{e^{-t/T}}{\Gamma(-j)}\left(\frac{T}{t}\right)^{1+j} v_\infty - e^{-t/T}\sum_{l=1}^{j}\frac{1}{\Gamma(1-j+l)}\left(\frac{T}{t}\right)^{l-j} v_l(0), \tag{6}$$

where, v_∞ is the maximum velocity permitted (leading car velocity), N is the total number of cars and $i = 1,2,\ldots,N$ and $\Gamma(\cdot)$ is the Euler gamma function.

Unfortunately, this model, and the solution given by (6), does not explain the empirically observed density waves. Therefore one has to introduce an additional time delay $\Delta t \sim O(1)$s, reflecting the finite reaction time of drivers. This yields the following stimulus-response model (also known as Pipes law) given by

$$\underbrace{\frac{dv_i(t+\Delta t)}{dt}}_{response} = \frac{1}{T}\underbrace{\left(v_{i+1}(t) - v_i(t)\right)}_{stimulos}. \tag{7}$$

Here, $1/T$ is the sensitivity to the stimulus. This equation belongs to the class of delay differential equations, and solutions can be unstable for $\Delta t > 0$. Indeed, for the above model, Chandler *et al*. (1958) [3] showed that a variation of individual vehicle velocities will be amplified under the instability condition $\Delta t/T > 1/2$. As a consequence, the nonlinear vehicle dynamics finally gives rise to stop-and-go waves, and also to accidents.

2.1.1 Statistical physics model

Statistical physics models for traffic flow are usually named Toda-Morse chains: these are models in which each car is a particle coupled to a "heat bath" and moving on a ring with particular asymmetrical springs among neighbours [1]. For the case under consideration we can construct a simple statistical physics model, starting by integrating Pipes rule (7), with $\Delta t = 0$. One gets

$$v_i = k(x_{i+1} - x_i) + k(c_{i+1} - c_i), \tag{8}$$

where c_i, are constants determine by the initial conditions and $k = 1/T$. If we assume that $v_i(0) = 0$ and that $x_i(0) = ih$ then it follows that $c_{i+1} - c_i = -h$. Introducing (8) in (7) yields

$$\dot{v}_i = k^2(x_{i+2} - 2x_{i+1} + x_i). \tag{9}$$

Defining the momentum $p_i = \dot{x}_i$ one can see that we have a Hamiltonian system with a "local" Hamiltonian

$$H_i = \frac{1}{2}p_i^2 - \frac{k^2}{2}(x_{i+2} - 2x_{i+1} + x_i)^2, \tag{10}$$

for which the equations of motion read

$$\dot{x}_i = \frac{\partial H_i}{\partial p_i} = p_i \tag{11}$$

$$\dot{p}_i = -\frac{\partial H_i}{\partial x_i} = k^2(x_{i+2} - 2x_{i+1} + x_i). \tag{12}$$

The total Hamiltonian is the given by

$$H(x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N; k) = \sum_{i=1}^{N} \frac{1}{2}p_i^2 - \frac{k^2}{2}(x_{i+2} - 2x_{i+1} + x_i)^2 \tag{13}$$

The statistical properties of this system are completed determined by the partition function

$$Z(\beta, k) = \int_{-L}^{L} \cdots \int_{-L}^{L} \int_{-v_\infty}^{v_\infty} \cdots \int_{-v_\infty}^{v_\infty} \exp(-\beta H(k)) dx_1 \ldots dx_N \, dp_1 \ldots dp_N, \tag{14}$$

where $L \sim O(10^3)\mathrm{m}$ the lengthscale, v_∞ the maximum permitted velocity and β an arbitrary parameter, the equivalent of the inverse temperature for the "gas" of cars. The partition function (14) permits to define a distribution probability function given by, taking $(\mathbf{x}, \mathbf{p}) = (x_1, x_2, \ldots, xN, p_1, p_2, \ldots, pN)$,

$$f(\mathbf{x}, \mathbf{p}) = \frac{\exp(-\beta H(\mathbf{x}, \mathbf{p}))}{Z(\beta, k)}. \tag{15}$$

The problem now rests in evaluating expression (14) for the partition function. Once this is done, it is possible to obtain the mean velocity, mean density, etc, as a function of separation time k^{-1} and the inverse temperature β. The canonical expression the mean value of any function $h = h(\mathbf{x}, \mathbf{p})$ is given by

$$< h >= \int_{-L}^{L} \cdots \int_{-L}^{L} \int_{-v_\infty}^{v_\infty} \cdots \int_{-v_\infty}^{v_\infty} h(\mathbf{x}, \mathbf{p}) f(\mathbf{x}, \mathbf{p}) dx_1 \ldots dx_N \, dp_1 \ldots dp_N, \quad (16)$$

thus allowing us to estimate the mean velocity $< v >$ as a function of the mean density $< \rho >$, which can be used to solve the equations of a macroscopic model.

2.2 MACROSCOPIC TRAFFIC MODELS

In contrast to microscopic traffic models, macroscopic ones are restricted to the description of the collective vehicle dynamics in terms of the spatial vehicle density $\rho(x,t)$ per lane and the average velocity $V(x,t)$ as a function of the freeway location x and time t. Macroscopic models have often been preferred to car-following models for numerical efficiency. Also, some favourable properties are (i) their good agreement with empirical data, (ii) their suitability for analytical investigations, (iii) the simple treatment of inflows from ramps, and (iv) the possibility of simulating the traffic dynamics in several lanes by effective one-lane models considering a certain probability of overtaking [1].

The oldest and still the most popular macroscopic traffic model goes back to Lighthill and Whitham (1955). Their fluid-dynamic model is based on the fact that, away from on or offramps, no vehicles are entering or leaving the freeway (at least if we neglect accidents). This conservation of the vehicle number leads to the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0. \quad (17)$$

Here

$$Q(x,t) = \rho(x,t) V(x,t), \quad (18)$$

is the traffic flow per lane, which is the product of the density and the average velocity. We may apply the so-called material derivative, moving with the cars,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}, \quad (19)$$

to rewrite equation (17) in the form

$$\frac{d\rho(x,t)}{dt} = -\rho(x,t) \frac{\partial V(x,t)}{\partial x}, \quad (20)$$

from which we conclude that the vehicle density increases in time $d\rho(x,t)/dt > 0$, where the velocity decreases in the course of the road $\partial V/\partial x < 0$, and vice versa.

Moreover, the density can never become negative, since $\rho(x,t) = 0$ implies $d_v\rho(x,t)/dt = 0$. Equation (17) is naturally part of any macroscopic traffic model. The difficulty is to specify the traffic flow $Q(x,t)$. Lighthill and Whitham assume that the flow is simply a function of the density

$$Q(x,t) = \rho V(\rho(x,t)) \geq 0. \tag{21}$$

Here, the fundamental (flow-density) diagram $Q(\rho)$ and the equilibrium velocity-density relation $V(\rho)$ are thought to be suitable fit functions of empirical data, for which there are many proposals. The first measurements by Greenshields (1935) had suggested a linear relation of the form

$$V(\rho) = V_0 \left(1 - \frac{\rho}{\rho_{jam}}\right), \tag{22}$$

which is still sometimes used for analytical investigations (this was the approach taken in the training session the week before!). Inserting (22) into the continuity equation (17), we obtain

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{23}$$

This is a nonlinear wave equation which describes the propagation of the kinematic waves with velocity

$$C(\rho) = \frac{dQ}{d\rho} = V(\rho) + \rho\frac{dV}{d\rho}, \tag{24}$$

$$= V_0 \left(1 - 2\frac{\rho}{\rho_{jam}}\right). \tag{25}$$

Note that $C(\rho)$ is the speed of the characteristic lines (*i. e.*, of local information propagation), which is density dependent. In contrast to linear waves, the characteristic lines intersect, because their speed in congested areas is lower. This gives rise to changes of the wave profile, that is, to the formation of "shock fronts", while the amplitude of kinematic waves does not change significantly.

3. HYBRID PARTIAL DIFFERENTIAL EQUATION

For the construction of a hybrid partial differential model the basic idea is to use a macroscopic model for the highway, but rely on some of the information from the microscopic model. As earlier we consider one lane and no overtaking. We use equation (17) and Pipes rule (7) [2,4]. It is convenient to write (see Figure 1) the position for the i car as $X_i(t) = ih$ and its velocity $U_i(t) = U(ih,t)$ where h is a typical spacing between cars ($\sim$ 10m).

Define the dimensionless variables

$$\tilde{t} = \frac{t}{t_0}, \qquad \tilde{x} = \frac{x}{L}, \qquad \tilde{X} = \frac{X}{L}, \qquad \tilde{U} = \frac{t_0}{L}U, \qquad \tilde{\rho} = L\rho, \qquad \tilde{u} = \frac{t_0}{L}u. \tag{26}$$

Using these new variables equation (17) reads

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial (\tilde{\rho}\tilde{u})}{\partial \tilde{x}} = 0, \tag{27}$$

and Pipes rule can be written in the form

$$\frac{\partial}{\partial t}\tilde{U}\left(\tilde{X}, \tilde{t} + \Delta\tilde{t}\right) = \frac{kt_0 h}{L}\left[\frac{U\left(\tilde{X} + h/L, \tilde{t}\right) - U\left(\tilde{X}, \tilde{t}\right)}{h/L}\right]. \tag{28}$$

Using the lengths values given and neglecting terms of order $O(10^{-2})$ and smaller one gets for (28)

$$\frac{\partial \tilde{U}(\tilde{X}, \tilde{t})}{\partial \tilde{t}} = C\frac{\partial \tilde{U}(\tilde{X}, \tilde{t})}{\partial \tilde{X}}, \tag{29}$$

where C is the only information left from Pipes rule and is given by

$$C = \frac{kt_0 h}{L} = O(1). \tag{30}$$

Dropping the ~, note that the variables X and x are not the same, *i. e.*, X is a Lagrangian variable and x an Eulerian variable, so we can not solve (27) and (29) directly. To relate X and x note that $dx/dt = u(x,t)$ and $x(0) = X$ defines $x(X,t)$. Also note that

$$\left.\frac{\partial}{\partial t}\right|_{X=cte} = \left.\frac{\partial}{\partial t}\right|_{x=cte} + u\left.\frac{\partial}{\partial x}\right|_{t=cte}, \tag{31}$$

and by conservation of the number of cars we have

$$0 = \frac{d}{dt}\int_X^{X+\delta X} \rho(x,t)\mathrm{d}x = \frac{d}{dt}\int_X^{X+\delta X} \rho(X,t)\frac{\partial x}{\partial x}\mathrm{d}X, \tag{32}$$

and so

$$\rho\frac{\partial x}{\partial X} = 1, \tag{33}$$

$$\int_{X_i}^{X_j} \rho dx = j - i. \tag{34}$$

Then the following applies

$$\frac{\partial U}{\partial t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \tag{35}$$

and consequently

$$\frac{\partial U}{\partial X} = \frac{1}{\rho}\frac{\partial u}{\partial x}. \tag{36}$$

So we can formulate a idea for BRISA: use the floating and the automatic data to estimate u and ρ at discrete points. Interpolate and use as initial data for solving the equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0, & (37)\\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} &= \frac{C}{\rho}\frac{\partial u}{\partial x}, & (38) \end{aligned}$$

for $0 < t < 10^2$, compare with the observed data (do some averaging) and then repeat…

This requires to solve (37) and (38) but this set of equations have very strange properties [5, 6, 7] namely that shocks and rarefaction waves coincide.

The main "peculiarity" is that we can rewrite both equations as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(u - \frac{C}{\rho}\right) &= 0, & (39)\\ \left(\frac{\partial}{\partial t} + \left(u - \frac{C}{\rho}\right)\frac{\partial}{\partial x}\right)u &= 0. & (40) \end{aligned}$$

Also the system is effectively linear in Lagrangian variables

$$\begin{aligned} \frac{\partial U}{\partial t} &= C\frac{\partial U}{\partial X}, & (41)\\ \frac{\partial \rho}{\partial t} + \rho^2\frac{\partial U}{\partial X} &= 0. & (42) \end{aligned}$$

This facts posses major theoretical challenges namely, how to characterize and solve such systems. Consider the Jacobian

$$\frac{\partial(x,t)}{\partial(X,t)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial t}{\partial X} \\ \frac{\partial x}{\partial t} & \frac{\partial t}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho} & 0 \\ u & 1 \end{vmatrix} = \frac{1}{\rho}. \tag{43}$$

Hence no shock can develop unless $\rho \to \infty$, *i. e.*, only with overtaking or accidents. Given $\rho(X,0) = \rho_0$, $U(X,0) = U_0$ we can integrate and obtain a solution in the Lagrangian frame with the form

$$\frac{1}{\rho(X,t)} = \frac{1}{C}\left[U_0(X+t) - U_0(X)\right] + \frac{1}{\rho_0(X)}. \tag{44}$$

And so $\rho \to \infty$ in finite time for suitable U_0.

4 CONCLUSION AND REMARKS

We have examined various aspects of traffic flow modelling. We first examined the simplest case of the Pipes rule assuming that the response time was null. We have obtained a explicit solution for this case. Unfortunately, the solution given does not explain the observed density waves usually found in traffic flow.

We therefore turned our attention to a statistical physics model. We wrote the Hamiltonian equations of motion based on Pipes rule and obtained the expression for the partition function and density distribution of cars assuming that each car is a particle coupled to a heat bath. We have also shown that the information that can be obtained from this approach could be included in a macroscopic model.

We then considered an hybrid partial differential equation approach. We have establish a connection between a microscopic follow-the-leader model based on ordinary differential equations and a macroscopic continuum model based on a conservation equation. We have found that the solutions of this system of partial differential equations have very strange properties, namely that shocks and rarefaction waves coincide in the same model. This fact posses major theoretical challenges namely, how to characterize and solve such systems.

Although the study group came to some results for the proposed onslaught, a particular exact solution, a statistical physics model and a hybrid partial differential equation, this last topic needs further research and a more sophisticated theoretically work is needed.

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